In the case of concavity of the graph of the unknown parameter

$$\Lambda(\theta) = \Lambda(0)\{1 + [-1 + \beta^{\frac{1}{n}}]\theta\}^{-n};$$

$$\theta = \frac{1 - \left\{\beta^{\frac{1-n}{n}} - \left(\frac{x-b}{c-b}\right)[1 - \beta^{\frac{1-n}{n}}]\right\}^{\frac{1}{1-n}}}{1 - \beta^{\frac{1}{n}}};$$

in the case of convexity

$$\Lambda(\theta) = \Lambda(0)\{1 + [-1 + \beta^{-n}]\theta\}^{\frac{1}{n}};$$

$$\theta = \frac{-\beta^{n} + \left\{1 - \left(\frac{x-b}{c-b}\right)[1 - \beta^{n+1}]\right\}^{\frac{n}{n+1}}}{1 - \beta^{n}}.$$

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CONSTRUCTION OF A REGULARIZED SOLUTION TO ONE INVERSE HEAT-CONDUCTION PROBLEM WITH RANDOM ERRORS IN THE INITIAL DATA

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An analysis is made of the statistical criterion for the choice of a regulation parameter in the reconstruction of the heat flux at the surface of a body from the temperature inside the body measured with a random error.

1. The determination of the heat flux at the surface of a body from the temperature field measured inside the body is a very common inverse boundary problem of heat conduction in the analysis of experimental results [1].

Let us consider an infinite plate with a thickness d which is thermally insulated on one side. The temperature field $t(x, \tau)$ at a depth x produced by a variable heat flux $g(\tau)$ entering through the boundary $x = d$ is determined by the integral equation [2]

$$\int_0^{\tau} h(x, d; \tau, \tau_1) g(\tau_1) d\tau_1 = t(x, \tau), \tag{1}$$

where $h(x, d; \tau, \tau_1)$ is the Green function for a plate of finite thickness. In the case of a nonzero initial temperature distribution $t(\xi, 0)$ the right side of (1) is written in the form [2]

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$$t'(x,\ \tau) = t(x,\ \tau) - \text{const}\int_0^d h(x,\ \xi;\ \tau,\ 0)\, t(\xi,\ 0)\, d\xi.$$

Because of the noise of the temperature sensor and the amplifiers, instead of the exact value of the temperature field $t(x, \tau)$ at the point x one is able to record a process $\tilde{t}(x, \tau) = t(x, \tau) + n(x, \tau)$, where $n(x, \tau)$ is the noise in the measurement of the temperature field. We make the following assumptions concerning the measurement noise: a) the process $n(x, \tau)$ is a steady random process with a zero mathematical expectation and does not depend on $t(x, \tau)$; b) the correlation function $K_n(\tau)$ for this random process is known. The latter assumption is satisfied very often in practice, since before conducting the experiment it is possible to record a process $\tilde{t}(x, \tau)$ whose pulsations are entirely due to the measurement noise and to calculate $K_n(\tau)$ for the required value of the argument using well-known methods.

Thus, the inverse heat-conduction problem (IHCP) under consideration can be formulated as follows: from the temperature field $\tilde{t}(x, \tau)$ recorded at the point x to determine the heat flux $g(\tau)$ entering the plate, which is the solution of a Volterra integral equation of the first kind (1).

Such a problem belongs to the class of incorrectly stated problems [3]. In the numerical solution of (1) the incorrectness entails the poor conditionality of the system of algebraic equations

$$Hg = \tilde{t}, \tag{2}$$

where g and $\tilde{t}$ are vectors of dimensionality N_g and N_t comprised of the values of $g(\tau)$ and $\tilde{t}(x, \tau)$ on some difference grid while H is an $(N_t \times N_g)$ matrix approximating the original integral operator, with the approximation error being negligibly small in comparison with the measurement noise. The poor conditionality of the system (2) is manifested in the characteristic "sawtoothed" form of its solution, which differs considerably from the true solution.

The Tikhonov regularization method, where the regularized solution is sought from the condition of the minimum of the smoothing functional, which ultimately leads to the solution of a well-specified system of equations, has obtained wide application in the solution of incorrectly stated problems. In our case the presence of a priori information on the statistical properties of the measurement noise allows us to use the methods of mathematical statistics both in the construction of the regularized solution and in the choice of the regularization parameter.

2. The regularized solution of the IHCP under consideration will be found from the condition of the minimum of the functional [4]

$$\Phi_\alpha[g,\ \tilde{t}] = \|\tilde{t} - Hg\|^2_{R^{-1}} + \alpha\|g\|^2_{\Gamma^{-1}}, \tag{3}$$

where R is an $(N_t \times N_t)$ correlation matrix with elements $\{R\}_{kj} = K_n(\tau_k - \tau_j)$; Γ is a positive-definite symmetrical $(N_g \times N_g)$ matrix assigned a priori and such that the quadratic form $\|g\|^2_{\Gamma^{-1}} = g^T\Gamma^{-1}g$ represents a finite-difference approximation of a functional reflecting the measure of smoothness of the unknown solution $g(\tau)$. The vector g_α at which the minimum of (3) is reached and which is determined from the system

$$(\alpha\Gamma^{-1} + H^T R^{-1} H) g_\alpha = H^T R^{-1}\tilde{t}, \tag{4}$$

where T is the transposition symbol, will be called the skeleton of the regularized solution of the IHCP.

An important problem arising in the construction of the skeleton g_α is the choice of the unknown regularization parameter α. From the point of view of the accuracy of the solution of Eq. (1) it is desirable to choose α so as to minimize some functional $\varepsilon(\alpha)$ of the error of the solution, characterizing the closeness of g_α to the pseudosolution $g^+ = (H^TH)^{-1}H^Tt$ of the system (2), which emerges in the role of the skeleton of the solution of the IHCP when the right side is assigned exactly. Mathematically this comes down to the solution of the extremal problem

$$\inf_{\alpha \geqslant 0} \varepsilon(\alpha) = \varepsilon(\alpha_0). \tag{5}$$

Henceforth we will call the parameter α_0 the Γ-optimal regularization parameter (RP), thereby emphasizing that α_0 minimizes $\varepsilon(\alpha)$ only with the adopted matrix Γ, while the operator $T(\alpha_0) = (\alpha_0\Gamma^{-1} + H^TR^{-1}H)^{-1}H^TR^{-1}$ is the Γ-optimal regularizing operator. In contrast to the well-known approaches to the choice of α, which permit one to construct solutions which are optimal in order of magnitude [5]; in the present report the Γ-optimal

TABLE 1. Errors in the Construction of Skeletons for the Solution of the IHCP

Noise level δ	Skeleton g_α		Skeleton g_{α_T}		Skeleton $\tilde{g}^+$	
	ε_1	ε_2	ε_1	ε_2	ε_1	ε_2
0,0187	0,0915	0,0462	0,1202	0,0591	8,98	7,17
0,0468	0,1002	0,0602	0,1581	0,0862	22,45	17,94
0,0937	0,1171	0,0861	0,2051	0,1261	44,91	35,88

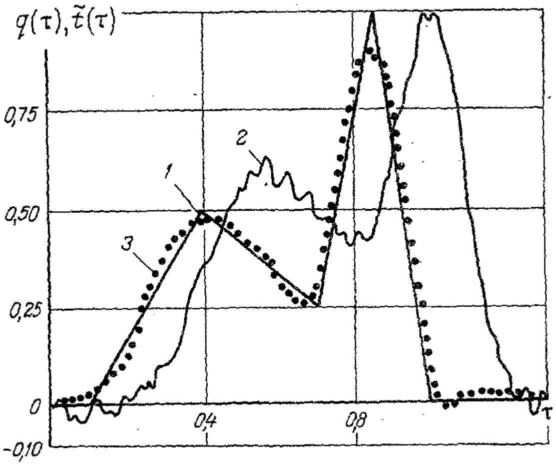

Fig. 1. Results of a numerical experiment on the reconstruction of the heat flux; 1) assigned heat flux $g(\tau)$; 2) temperature $\tilde{t}(\tau)$ measured with a noise $\delta = 0.046$; 3) regularized skeleton g_α constructed with $\alpha = \alpha_A$.

value of the RP is calculated on the basis of the testing of statistical hypothesis concerning the covariation matrix of the discrepancy vector.

3. As the criterion of accuracy in the construction of the regularized skeleton of the solution of the IHCP we take the root-mean-square error $\varepsilon(\alpha) = M[\|g_\alpha - g^+\|^2]$, where M is the mathematical expectation operator. Introducing the matrix $V_\varepsilon(\alpha) = M[(g_\alpha - g^+)(g_\alpha - g^+)^T]$ into the analysis and using $\varepsilon(\alpha) = \mathrm{Sp}[V_\varepsilon(\alpha)]$, where Sp[A] is the spur of the matrix A, the problem (5) can be reduced to the extremal problem

$$\varepsilon(\alpha_0) = \inf_{\alpha \geqslant 0} \mathrm{Sp}\,[T(\alpha) V_{\tilde{t}} T^T(\alpha) - 2V_g H^T T^T(\alpha) + V_g], \tag{6}$$

where $V_{\tilde{t}} = M\,[\tilde{t}\,\tilde{t}^T]$; $V_g = g^+(g^+)^T$. The solution of the problem (6) allows one to obtain the sufficient condition for the Γ-optimality of the value of the RP, which has the form of the identity

$$T(\alpha_0) V_{\tilde{t}} = V_g H^T. \tag{7}$$

Since V_g is not known, however, this identity cannot be used for the construction of the Γ-optimal regularizing operator. This difficulty can be overcome in the following way.

Let us determine the covariation matrix $V_e(\alpha) = M[e_\alpha e_\alpha^T]$ of the discrepancy vector $e_\alpha = \tilde{t} - Hg_\alpha$ with $\alpha = \alpha_0$. We can show that from (7) we immediately get the identity

$$V_e(\alpha_0) = R(R + \alpha_0^{-1} H\Gamma H^T)^{-1} R, \tag{8}$$

which represents the sufficient condition for the Γ-optimality of the RP.

The identity (8) is essentially the criterion for the Γ-optimality of the RP, and in contrast to (7) its use does not require a priori information about g^+. However, to test (8) one must know the matrices standing on both sides of (8). The difficulty arising in this stage consists in the impossibility of calculating the matrix $V_e(\alpha)$ from one realization of the random discrepancy vector. Therefore, to test (8) one must apply the methods for testing statistical hypotheses on the equality of covariation matrices, which are widely used in multivariate statistical analysis [6].

As the null hypothesis A_0 we take the assumption that $V_e(\alpha) = R(R + \alpha^{-1}H\Gamma H^T)^{-1}R$, while as the alternative A_1 we take $V_e(\alpha) \neq R(R + \alpha^{-1}H\Gamma H^T)^{-1}R$. The value of the RP at which the null hypothesis is accepted, i.e.,

$$V_e(\alpha_A) = R(R + \alpha_A^{-1}H\Gamma H^T)^{-1}R, \tag{9}$$

using the chosen criterion F, is designated as α_A. One can show that $\varepsilon(\alpha_A) < \inf_{\alpha \in \{\alpha_1\}} \varepsilon(\alpha)$, where $\{\alpha_1\}$ is the set of values of $\alpha \geq 0$ for which the hypothesis A_1 is accepted. Thus, the value of α_A is the Γ-optimal RP for the set $\alpha_A U\{\alpha_1\}$ and for the construction of g_α it only remains to find the α at which the hypothesis A_0 is accepted.

The simplest algorithm for finding α_A is based on the testing of the distribution of the quadratic form $\rho(\alpha) = e_\alpha{}^T R^{-1} (R + \alpha^{-1}H\Gamma H^T)R^{-1}e_\alpha$, which, with the acceptance of the hypothesis A_0, must conform to a χ^2 distribution with N_t degree of freedom. Without repeating the arguments expounded in [7], we only present the final entry of the calculating procedure of this algorithm, which has a recurrent form relative to $\gamma = 1/\alpha$:

$$\gamma_{k+1} = \gamma_k + \frac{R(\gamma_k)}{R'(\gamma_k)}\left[1 - \frac{R(\gamma_k)}{N_t}\right], \quad \gamma_0 \ll 1, \tag{10}$$

where $R(\gamma) = \rho(1/\gamma)$, while the derivative $R'(\gamma)$ is determined numerically: $R'(\gamma) = [R(\gamma + \Delta\gamma) - R(\gamma - \Delta\gamma)]/2\Delta\gamma$. The process of finding $\alpha = 1/\gamma$ ceases when the value of $R(\gamma)$ falls within the interval $\theta_{N_t}(a) = [\vartheta_{N_t}(a/2), \vartheta_{N_t}(1 - a/2)]$, where $\vartheta_{N_t}(a/2) - a/2$ is the quartile of a χ^2 distribution with N_t degrees of freedom, which corresponds to the acceptance of the hypothesis with an error of the first kind equal to a. The procedure (10) has a high rate of convergence to the value α_A and the number of iterations needed for $R(\gamma)$ to fall within the interval $\theta_{N_t}(a)$ usually does not exceed five.

We note that the algorithm for choosing α on the basis of the testing of the hypothesis A_0 guarantees the convergence of g_α to g^+ in the root-mean-square metric [7], i.e., $\lim M[\|g_\alpha - g^+\|^2] = 0$ as $Sp[R] \to 0$.

To estimate the statistical characteristics of the error in the construction of the skeleton of the solution of the IHCP it is convenient to introduce the displacement vector $b_\alpha = M[g_\alpha] - g^+$ and the correlation matrix $C_\alpha = M[g_\alpha - M[g_\alpha]) (g_\alpha - M[g_\alpha])^T]$ [7].

The algorithm described above for the construction of the regularized skeleton was placed at the basis of a program for the solution of Eq. (1) which is used successfully at the Institute of Thermophysics, Siberian Branch of the Academy of Sciences of the USSR, and other organizations.

4. Let us present the results of a numerical experiment. A measurement noise with a relative error $\delta = \left(\sum_j n^2(j)\right)^{1/2} / \left(\sum_j t^2(j)\right)^{1/2}$ was imposed on a temperature field $t(\tau)$, measured exactly at the point x and corresponding to an assigned heat flux $g(\tau)$. The skeleton g_α was constructed from the vector $\tilde{t}$ thus obtained. The skeleton g_α for $\delta = 0.046$ is presented in Fig. 1. Despite the discontinuities in the derivatives of the function $g(\tau)$, which makes it "poor" for reconstruction, the skeleton g_α coincides rather well with $g(\tau)$. The relative values

$$\varepsilon_1 = \max_j |\varepsilon_\alpha(j)| / \max_j |g^+(j)|; \quad \varepsilon_2 = \left(\sum_j \varepsilon_\alpha^2(j)\right)^{1/2} / \left(\sum_j (g^+(j))^2\right)^{1/2}$$

of the error $\varepsilon_\alpha(j) = g_\alpha(j) - g^+(j)$ for g_α and for the skeleton $g_{\alpha T}$, constructed using Tikhonov's first-order regularizing operator with α chosen in accordance with the discrepancy principle, i.e., from the condition $\|\tilde{t} - Hg_{\alpha T}\|^2 = N_t\sigma_{nn}^2$, where σ_{nn}^2 is the dispersion of the measurement noise, are presented in Table 1. The errors in the construction of the skeleton $\tilde{g}^+ = (H^TH)^{-1}H^T\tilde{t}$, which represents the unregularized solution of the IHCP, are also presented here. It is seen that the use of regularization methods considerably increases the accuracy of the solution of the IHCP. The smaller error in the construction of the skeleton g_α can be explained by the allowance for the statistics of the measurement noise in the construction of the regularized solution of the IHCP and the choice of the regularization parameter.

In conclusion, we note that the approach to the choice of the regularization parameter discussed in the report can be used in the solution of poorly specified systems of algebraic equations for which the right sides are given with an error having a random nature.

NOTATION

$g(\tau)$, heat flux at surface of plate; $t(x, \tau)$ temperature at the point x; $n(x, \tau)$, noise of temperature measurement; $\tilde{t}(x, \tau)$, temperature measured at the point x; τ, time; $\tilde{t}$ and g, vectors composed of the values of the functions $\tilde{t}(x, \tau)$ and $g(\tau)$; H, matrix approximating the original integral equation; α, regularization parameter.

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NUMERICAL ALGORITHM FOR THE SOLUTION OF LINEAR TWO-DIMENSIONAL INTEGRAL EQUATIONS OF THE FIRST KIND

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A numerical algorithm is proposed for the solution of two-dimensional integral equations of the first kind, to which some inverse problems of heat conduction are reduced.

It is known that many problems of practical importance in the analysis of experimental results lead to the solution of a linear integral equation of the first kind in a rectangular region (for example, thermophysical problems of the determination of the heat flux to axisymmetric and plane bodies from the assigned time dependence of the temperature at part of the boundary of the region, the geophysical problem of the interface between two media with different densities, and others):

$$Af \equiv \iint_{D_1} K(x, y, s, t)\, f(s, t)\, ds dt = \psi(x, y),$$
$$x, y \in D = \{0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1\}; \quad s, t \in D_1(x, y) \subset D. \tag{1}$$

The problem of the solution of such an equation is, generally speaking, incorrectly stated. If the solution of Eq. (1) is unique for an assigned right side $\psi(x, y)$ then the solution $f(s, t)$ can be obtained by the regularization method proposed by Tikhonov [1, 2]. In accordance with this method an approximate solution $f^{\alpha}(s, t)$ is defined as a function yielding the minimum of the functional

$$M^{\alpha}[f, \psi] \equiv \iint_D [Af - \psi]^2 dxdy + \alpha \iint_D \{p_1 f^2 + p_2 f_s^2$$
$$+ p_3 f_t^2 + \beta [p_4 f_{ss}^2 + p_5 f_{tt}^2 + p_6 f_{st}^2]\} dsdt, \tag{2}$$

in which the value of the regularization parameter α must conform with the level of the root-mean-square error δ of the right side. In the functional (2) the quantities $p_i(s, t) > 0$ ($i = 1, 2, \ldots, 6$) are assigned functions and $\beta = 0$ or 1 in first- or second-order regularization, respectively.

In [3] an algorithm was proposed for the solution of the variational problem (2) for the one-dimensional equation (1), based on the approximation of the unknown solution by cubic splines [4]. The effectiveness of the algorithm, verified on the problem of solving the Abel equation [3] and the problem of reconstructing a distribution function [5], is explained to a considerable extent by the properties of the convergence of cubic splines

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